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IMPOSING ‘NO’ BOUNDARY CONDITION AT OUTFLOW: WHY DOES IT WORK?

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SUMMARY

In recent work on outflow boundary conditions for Navier–Stokes equations by Papanastasiou *et al.* (*Int. j. numer. methods fluids*, **14**, 587–608 (1992)) a choice has been proposed which can formally be described as imposing no boundary condition at all. This of course leads to an underdetermined problem at the level of the partial differential equations. However, it yields a well-defined problem at the discrete level and it has been documented that this choice of outflow conditions performs in a way which is superior to more ‘standard’ artificial boundary conditions. In this paper we analyse a one-dimensional model problem. We shall show that the ‘free’ boundary condition of Papanastasiou *et al.* actually imposes an effective boundary condition. This effective boundary condition is identified and its advantages are discussed.

KEY WORDS: open boundaries; extrapolation conditions; finite elements

1. INTRODUCTION

Numerical solutions of flow problems frequently lead to the need to impose artificial boundary conditions at ‘open’ boundaries which are introduced by truncation of the flow domain. The choice of such boundary conditions poses a major problem. Recently, a ‘free’ boundary condition was proposed by Papanastasiou *et al.*[1] Interestingly, this condition can formally be described as imposing no boundary condition at all!

We now describe how the free boundary condition is introduced. In the finite element method the solution is approximated by a linear combination of selected shape functions and the differential equation is approximated by setting its integral against each shape function equal to zero. These discretized equations are then integrated by parts. For shape functions supported in the interior, this integration by parts is straightforward, but for shape functions associated with boundary nodes, there are boundary integrals arising from the integration by parts. These boundary integrals are dealt with in one of two ways. If the boundary condition is essential, the equation associated with the boundary node is dropped and replaced by the prescribed boundary condition. If the boundary condition is natural, then the boundary integral from the integration by parts is prescribed. The suggestion in Reference 1 for the open boundary is simply not to do the integration by parts at all for those equations associated with nodes on the outflow boundary. Formally, this appears to impose no boundary condition if the order of the shape functions is at least equal to that of the differential equation. Otherwise, a boundary condition is imposed which is obtained simply by setting the highest-order derivatives equal to zero.[2]

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At the level of the partial differential equation it is of course impossible to simply drop a boundary condition. However, the discrete problem in Reference 1 yields perfectly well-defined solutions which appear to be better than those obtained with more 'classical' choices of boundary conditions. This makes it of interest to understand what the effect of the free boundary condition is. Indeed, Papanastasiou *et al.*[1] observe that in the Stokes limit the free boundary condition appears to be equivalent to natural, stress-free conditions. This suggests that actually the free condition may mask a hidden 'effective' boundary condition.

In this paper we shall study a one-dimensional model problem where we demonstrate that this is indeed the case. The equation is

$$u_t = \epsilon u_{xx} - cu_x - bu + f, \tag{1}$$

posed for $x \in (0, L)$ and $t > 0$, where $f(x, t)$ is a given function and ϵ, c and b are non-negative constants. For simplicity we consider a Dirichlet condition at the left endpoint:

$$u(0, t) = 0. \tag{2}$$

The right endpoint is treated as an 'open' boundary. The same kind of model problem is also discussed in Reference 2. It is pointed out there that if piecewise linear elements are used for discretization, then the free boundary condition is equivalent to $u_t = -cu_x - bu + f$, i.e. it is obtained from the differential equation by setting the second derivative equal to zero. However, no boundary condition was identified for quadratic elements, it was pointed out, however, that the case $b = c = 0$ leads to an ill-posed problem and it was conjectured that other cases might lead to an implicit boundary condition (called the 'fuzzy' boundary condition in Reference 2). In this paper we consider discretization by quadratic three-node elements.[3,4] We shall show that the 'free' boundary condition of Reference 1 is effectively equivalent to imposing the boundary condition

$$c(u_t + cu_x + bu - f) = \epsilon(-u_{xt} - bu_x + f_x). \tag{3}$$

This condition is obtained by differentiating (1), setting u_{xxx} to zero and substituting u_{xx} from (1). We note that (3) will not work if we consider the time-independent problem with $b = c = 0$; indeed, in this case the free boundary condition cannot work, since every linear function satisfies the equation even at the discrete level (see the remarks at the end of Section 6 in Reference 1).

The procedure of setting $u_{xxx} = 0$ can be viewed as a special case of 'extrapolation' conditions. In the present case the condition $u_{xxx} = 0$ is not introduced explicitly but is hidden in the discrete formulation. There are a number of results in the literature where boundary conditions for sufficiently high-order derivatives of the solution are discussed. We refer e.g. to Reference 5, where the boundary condition $\Delta u = 0$ is used for an advection–diffusion equation and error estimates for high Peclet number are derived. In Reference 6, outflow boundary conditions for the Navier–Stokes equations are considered which consist of setting higher-order derivatives of the velocities equal to zero. It is demonstrated that such a procedure is effective in suppressing high-Reynolds-number boundary layers. In addition, the stability of various combinations of inflow and outflow conditions is discussed. In Reference 7, error estimates (for the error resulting from erroneous boundary data) are given for extrapolation outflow conditions. Since the 'free' boundary condition is effectively equivalent to an extrapolation condition, it offers the same advantages which have been documented for these conditions, such as the absence of boundary layers. We shall discuss this point in the concluding section.

2. DISCRETIZATION BY PIECEWISE QUADRATIC ELEMENTS

We shall be concerned only with spatial discretization of (1) and continue to consider time as a continuous variable. The discretization will be by three-node piecewise quadratic elements; see e.g. Reference 3 or 4. We divide the interval [0, L] into N subintervals of length $L/N = h$; let $x_i = iL/N$. Associated with each node x_i, we have the basis function $\phi_i(x) = \phi(x - x_i)$, where

$$\phi(x) = \begin{cases} 1 + 3x/h + 2(x/h)^2 & \text{if } -h \leqslant x \leqslant 0, \\ 1 - 3x/h + 2(x/h)^2 & \text{if } 0 \leqslant x \leqslant h, \\ 0 & \text{if } |x| \geqslant h. \end{cases} \tag{4}$$

For $i = 0$ and N we simply ignore the part of ϕ_i which is outside the interval [0, L]. Moreover, for each interval we have the internal node $x_{i+1/2} = (x_i + x_{i+1})/2$ and the associated basis function $\phi_{i+1/2}(x) = \psi(x - x_i)$, where

$$\psi(x) = \begin{cases} x/h - (x/h)^2 & \text{if } 0 \leqslant x \leqslant h, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Equation (1) with boundary condition (2) is then discretized by setting

$$u = \sum_{i=1}^{2N} \alpha_i \phi_{i/2}(x) \tag{6}$$

and requiring that

$$\int_0^L (u_t - \epsilon u_{xx} + cu_x + bu - f)\phi_i \, \mathrm{d}x = 0 \tag{7}$$

for $i = \frac{1}{2}, 1, \ldots l, N$. Here u_{xx} needs to be interpreted in the sense of distributions, since u_x may have jumps at element boundaries. For all i other than N we can integrate by parts to find

$$\int_0^L [(u_t + cu_x + bu - f)\phi_i + \epsilon u_x \phi_i'] \mathrm{d}x = 0. \tag{8}$$

However, for $i = N$ this results in a boundary term, since $\phi_N(L) \neq 0$:

$$\int_0^L [(u_t + cu_x + bu - f)\phi_N + \epsilon u_x \phi_N'] \mathrm{d}x - \epsilon u_x(L, t)\phi_N(L) = 0. \tag{9}$$

The 'free' boundary condition of Reference 1 consists in simply evaluating $u_x(L, t)$ from (6) without imposing any additional conditions. This is of course equivalent to simply keeping the equation in the form (7).

We now focus on (7) for the last two values of i, i.e. $i = N - \frac{1}{2}$ and $io = N$. For those values we get

$$\int_0^h [u_t(L-\xi, t) - \epsilon u_{xx}(L-\xi, t) + cu_x(L-\xi, t) + bu(L-\xi, t) - f(L-\xi, t)][1 - 3\xi/h + 2(\xi/h)^2]\mathrm{d}\xi,$$
$$\int_0^h [u_t(L-\xi, t) - \epsilon u_{xx}(L-\xi, t) + cu_x(L-\xi, t) + bu(L-\xi, t) - f(L-\xi, t)][\xi/h - (\xi/h)^2]\mathrm{d}\xi. \tag{10}$$

We note that since the integrations extend only over a single element, we can interpret all derivatives here in the classical sense. We subtract the two equations from each other:

$$\int_0^h [u_t(L-\xi,t) - \epsilon u_{xx}(L-\xi,t) + cu_x(L-\xi,t) + bu(L-\xi,t) - f(L-\xi,t)][1 - 4\xi/h + 3(\xi/h)^2]\mathrm{d}\xi. \tag{11}$$

We now integrate by parts, but we shift derivatives onto u rather than away from it. This yields

$$\int_0^h [u_{xt}(L-\xi,t) - \epsilon u_{xxx}(L-\xi,t) + cu_{xx}(L-\xi,t) + bu_x(L-\xi,t) - f_x(L-\xi,t)] \times [\xi/h - 2(\xi/h)^2 + (\xi/h)^3]\mathrm{d}\xi = 0. \tag{12}$$

We now note that u is a quadratic function on the interval $[L-h, L]$ and hence $u_{xxx} = 0$. Moreover, the weight function

$$\xi/h - 2(\xi/h)^2 + (\xi/h)^3 \tag{13}$$

is positive for $\xi \in (0, h)$. Hence (12) says that a weighted average of

$$u_{xt} + cu_{xx} + bu_x - f_x \tag{14}$$

vanishes in the last element. We can therefore claim that the 'free' boundary condition is equivalent to the effective boundary condition

$$u_{xt} + cu_{xx} + bu_x - f_x = 0. \tag{15}$$

We can now use equation (1) to replace u_{xx} in (15). This leads to (3).

3. DISCUSSION OF THE EFFECTIVE BOUNDARY CONDITION

We start by discussing some limiting cases. First let us consider the case where $c = 0$ and $f = 0$. In that case, (3) takes the form

$$u_{xt} = -bu_x. \tag{16}$$

That is, we have the natural boundary condition $u_x = 0$ in the time-independent case, while in the time-dependent case we converge to the natural boundary condition as $t \to \infty$. We note that the equation $u_t = \varepsilon u_{xx} - bu$ can be viewed as a prototype for the two-dimensional problem

$$u_t = \epsilon(u_{xx} + u_{yy}) \tag{17}$$

if we can separate variables and use normal modes in the y-direction. We note that Papanastasiou *et al.*[1] indeed appear to find a natural boundary condition in the case of the Stokes problem.

Next we consider the case of high Peclet number and $\epsilon \to 0$. If we actually set $\epsilon = 0$, then (3) simply coincides with the differential equation itself. Moreover, the right-hand side of (3) represents the right first-order correction. If ε is small, then we can differentiate (1) and, to within an error of order ε, we have

$$u_{xt} = -cu_{xx} - bu_x + f_x. \tag{18}$$

If we solve this for u_{xx} and substitute into (1), we obtain precisely (3). Hence solutions of (1) which have no boundary layer at $x = L$ should satisfy (3) to within an error of order ϵ^2. This makes (3) a natural candidate for a boundary condition to be imposed if boundary layers are to be avoided. Indeed, the absence of boundary layers at high Reynolds number is one of the most striking successes reported in Reference 1.

Finally let us consider (1) on the semi-infinite interval $(0, \infty)$ and let us assume that f and u approach some asymptotic limit as $x \to \infty$. Clearly, the preferred conditions to be imposed at $x = L$ would be those which are consistent with this asymptotic limit. If u approaches a limit at large x, we should have $u_t + bu - f$, u_x and u_{xx} approximately equal to zero at large x, which is consistent with (3). Moreover, if the approach to the asymptotic limit is slow, then u_x will dominate over u_{xx} for large x and we are back to the high-Peclet-number limit which we have already discussed. Hence the boundary condition (3) does, under these conditions, reflect the asymptotic behaviour of the true solution on the semi-infinite interval. We note that this does not apply if the solution asymptotes to a periodic rather than a constant state at infinity. Hence the method should do less well in these cases. We note that in Reference 1 the asymptotic state was always 'fully developed' flow.

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